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THE LAW OF COMPARATIVE JUDGMENT: THEORY AND IMPLEMENTATION

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ABSTRACT

A method for obtaining a perceptual ranking (scaling) for defining texture measures is described. This method can be used to scale the relative visual differences among a set of texture pairs. This perceptual ranking is called the law of comparative judgment.

1- INTRODUCTION

A theoretical method for defining texture measures was defined by Conners and Vasquez [1,2]. The use of this method requires a perceptual ranking which can be used to rank the relative visual differences among a set of texture pairs. This report describes the theoretical development and an implementation of such a perceptual ranking. This perceptual ranking is called the law of comparative judgment (LCJ). It was developed by Thurstone [3]. It allows n things to be ranked (scaled) based on pairwise responses obtained over all possible combinations of n things taken two at a time.

The n things to be ranked in the measurement definition problem are n texture pairs. The scaling determines the relative discriminability of the pairs, i.e., which pair is the most visually distinct, which is the next most visually distinct, etc. The pairwise responses are experimental data obtained by showing subjects two texture pairs at a time. Each subject is asked to independently give his opinion as to which of the two is the more visually distinct texture pair.

In what follows, a background on perceptual ranking will be given and the reasons for selecting the LCJ will be presented. Then a theoretical development of the LCJ is given in sections 3 and 4 which develops the mathematical

equations necessary to implement the method. Documentation for the software based there mathematical equations are presented in section 5. Finally in section 6 some samples runs are given.

2. -BACKGROUND

For the measurement definition problem, a perceptual ranking (scaling) is neccesary. A psychological experiment has to be performed in order of obtain this scaling. This section presents a brief description on psychological scaling methods Psychological scaling methods are procedures for constructing scales for the measurement of psychological attributes.

The measurement of observers' responses to stimuli grew up in what is called psychophysics. Psychophysics was defined by Gustav Theodor Fechner as "an exact science of the functional relations of dependency between body and mind." [4] As developed by Fechner [5], psychophysics includes both the measurement of sensory attributes and the quantification of perception, in order to correlate these psychological scales with physical measurements of the stimuli. He suggested that the sensation intensity was proportional to the logarithm of the stimulus intensity.

L. L. Thurstone [3] pointed out that there were two classes of psychophysical methods. One class required that the experimenter be able to obtain some physical measurement of the stimulus, and to control this measurement for purposes of his experiment. Examples of this class are the method of

average error and the method of minimial changes [6]. A second class could be readily applied in cases where precise measurement and controlled variation of the physical characteristics of the stimuli were not posible. The method of paired comparisons is a second class example.

In his overview of psychophysical scaling methods, F. Nowell Jones [7] divided the psychophysical scaling in two methods: the direct methods and the indirect methods. The direct methods requires that judgments be made either according to some predetermined ratio given by the experimenter, or made in terms of real numbers. Thus the data collection involves a judgment in terms of a scale external to the stimuli themselves. Example of methods that belong to this group are methods involving judgment of assigned intervals, fractionation methods, methods of multiple production or multiple judgments, the constant sum or ratio partition method, and magnitud estimation. The "indirect," or Fechnerian methods, seem to be so called because considerable statistical manipulation is required for the constructions of a measurement scale. Actually, data collection is "direct" for these methods, because what is required is direct judgments of differences among stimuli. Methods that belong to this group are The law of comparative judgments and categorical judgments. The best method for our perceptual scaling belongs to the indirect method because the

needs of direct judgments of differences among texture pairs (stimuli). For this reason, let discuss or describes the indirect methods. For the description of the different methods for the direct case and also indirect cases the reader can be referred to the following references [4,6-13].

The original Fechnerian idea was based on what he called Weber's law [5]. That is, if discrimination requires a constant proportional increase in the stimulus, a function $dr/R=k$, may be written for some probability of discrimination, and if one assume that this relationship holds for very small increments, one may regard this formula as giving the relationship between the stimulus and subjective increment. This leads to the statement that $S=K \log R$. Now, if what one needs is the Weber fraction (k), and if one assumes that it is constant over a long range of stimuli, any psychophysical method that yields a measure of discrimination will give us a subjective scale. In practice, this is not done. The above is known as classical psychophysic method. The law of comparative judgment belongs to a group of methods known as psychological-scaling. It differs from the traditional or classical psychophysical method in that at the end results are no values on physical scales but are on psychological scale.

Modern work on indirect scaling was begun by Thurstone

with the publication of the LCJ. The idea was that a stimulus - whether physical or otherwise- gives rise to a hypothetical discriminal process within the subject which, for various random reason, varies from presentation to presentation of the same stimulus. The LCJ can be considered as a probabilistic model. This model assumes that the scale positions belonging to the psychological objects are themselves stochastic. Then the scale position does not have a fixed value but is regarded as a stochastic or random variable with an associated probability density function. An assumption is needed to form the density function. A popular assumption is that the scale positions are normally distributed.

The most usual method of obtaining data for use in scaling according to the LCJ is by means of pair comparisons. The main advantage of the method of pair comparisons is that it yields an estimate of subjective distance over the range of whatever stimuli are used. It is possible to use stimuli that cannot be arranged on an objective dimension. One need not know in advance which stimuli lie next to each other subjectively. There are two main disadvantages. First, there must be some degree of confunsion between adjacent stimuli since, if not, we have seen that no estimate of distance is possible. The second disadvantage is that the method requires a good many judgments for the amount of

information extracted. Other scaling method has been developed using the method of paired comparison or a variation of the method to try to overcome the above disadvantage. But all of these methods are very restrictive to be used in our definition problems. These methods are the composite standard [13] and the proposed by Guttman [14].

The other method for collection of data for comparative judgment is the method of rank order. In this method, the subject is asked to arrange a set of stimuli in accordance with the amount of some property. This method differs psychologically from the pairs comparisons in the stimuli are all presented at the same time and hence the judgment are made in the context of the total range, whereas the total range enters into pair comparisons only by way of some memory process. To derive, a scale from rank data is ordinarily accomplished in one of two ways. The first assume that the stimuli were drawn from a population of stimuli that is normally distributed with respect to the property of interest. The second method is derived from the LCJ [13]. The advantage of this method is less time consuming that the method of pair comparison. The method of categorical judgment was developed by Togerson [4]. The subject is presented with a succession of stimuli that he is to place in appropiate category, where the experimenter has determined the number of categories to be used. This method is no

appropriated for our experiment.

Of all the techniques mentioned, the best technique that seems suitable for our experiment purpose is the LCJ. The data collection will follow the method of paired comparison. The LCJ is applicable not only to the comparison of physical stimuli intensities but also to qualiltative comparative judgment such as those of excellence of specimens in an educational scale, and it has been applied in the measurement of such psychological values as a series of opinions on disputed public issues. Also, it has been used for scaling social values, nationality preferences, temperature-moisture, the lifted-weight experiment, etc. More recently this law was used by Tamura et al. [15] to construct a psychometric prototypes with which the measures computed from a set of texture could be compared. In the next section, a complete discussion on the LCJ is given.

3. -THE LAW OF COMPARATIVE JUDGMENT (LCJ)

A. The Psychological Theory

Thurstone [3] postulates an one-dimensional psychological scale onto which stimuli (texture pairs) are mapped. The nature of this scale is left unspecified: it may be psychic, physiological or both. The concepts is as follow: each time a stimulus (texture pairs) is presented it is presumed to be represented by a point along the psychological scale. The location of the point is determined by an unknown discriminal process by which the organism identifies, distinguishes, discriminates, or reacts to stimuli. Because of the uncertain nature of a person's perceptual state, the same stimulus does not always excite the same discriminal process. It is assumed that repeated occurrences of a stimulus produce a distribution called a discriminal dispersion of such processes along the psychological scale. A normal distribution is usually assumed. These random events will tend to describe a normal distribution around a mean. The mean is associated with the scale value of the stimulus, and the standard deviation is interpreted as the unit of measurement along the internal scale.

For convenience, Thurstone represented each stimulus on a hypothetical psychological continuum by the single

discriminal process corresponding to the mean of its discriminal dispersion. By using the standard deviation of the discriminal dispersions as units of measure, scale value are then established. Thus the means of the discriminal dispersions are the scale values measured on an interval scale in units of standard deviation. Pairs of stimuli are represented for judgment to obtain an empirical estimates of the distance along the psychological scale separating each stimulus from every other one.

Lets consider the theroretical distribution of discriminal processes for any two stimulus j and k as show in figure 1. These stimuli are associated on the psychological scale with theirs respective normal discriminal dispersion with means uk and uj and standard deviation σk and σj.

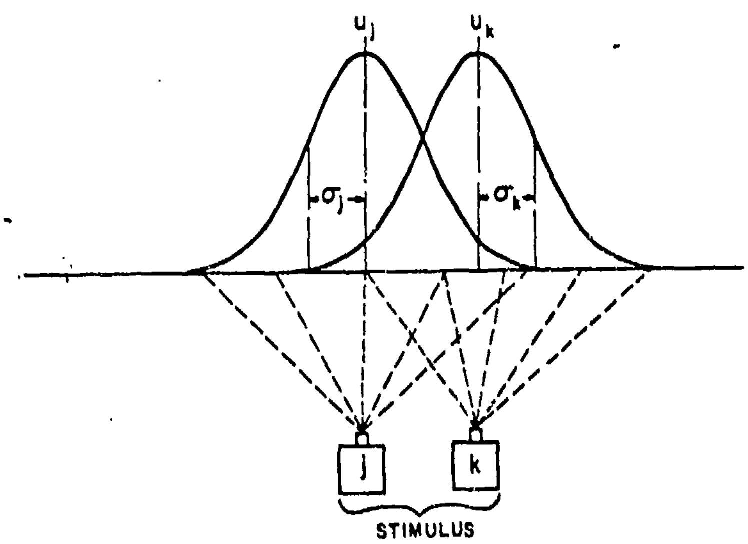

Figure 1. Discriminal dispersion for stimuli j and k. The means of the hypothetical distributions are u_j and u_k with standard deviation $\sigma_j = \sigma_k$.

If the two stimulus were presented together to an observer on a large number of occasions, each would excite a discriminal process on each presentation. i.e., a point along the scale. These two discriminal proceses are compared. On those occasion when the process associated with k is greater than the process for, j the observed will judge k to be greater than j, an vice versa. Since the two distributions are normal, no value for a process is impossible. The two distribution will overlap, and theoretically a stimulus will not be judged greater than another on 100% of the trials. In figure 1, it is clear that k will be judged greater than j on most occasions since most of the distribution for k has higher values that the one for j. But assuming random sampling from each distribution, we can expect a reversal once in a while (j > k).

In the analysis of stimulus pairs, one does not directly measure the variance and means of individual discriminal dispersions. Instead, one is receiving information on the distribution generated by all possible pairs of processes selected from the two discriminal dispersions. One needs to have the appropriate assumptions in which the information on the individual dispersions is directly translated into information on the distribution of differences and vice versa.

Figure 2 shows the distribution associated with four stumuli: 1,2,3, and 4. The scale value for stimulus 1 is u1, of stimulus 2 is u2,etc.

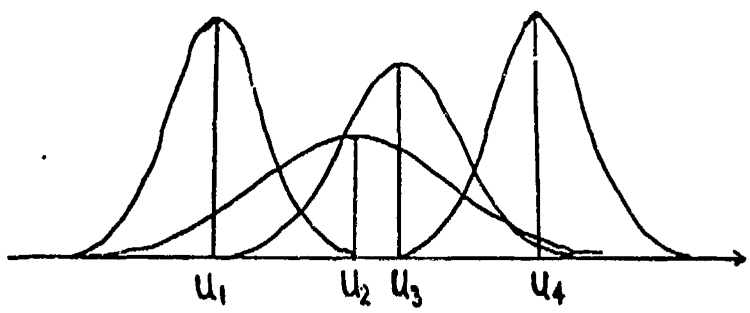

Figure 2. Distributions of discriminal processes associated with four stimuli.

B. The Law of Comparative Judgment

One wishes to estimate the distance between stimuli and use this information to locate the stimuli relative to each other along one dimensional psychological scale. Lets assume that each pair is associated with a single hypothetical distribution of differences generated by pairing all possible

discriminal processes in j with all discriminal processes in k. Therefore, the subject uses the differences in the magnitude of discriminal processes to make a decision concerning the dominance of one stimulus over another. From statistics, the difference between the means of two normal distribution is equal to the mean of their differences. Then, to find the differences in scale values for two stimulus (k and j), the mean of their distribution of differences has to be found. This mean can be measured arbitrarily from a point representing those cases where the difference between two discriminal processes, one for each stimulus, is 0. Lets locate, for convenience, the zero point as the mean discriminal process for the stimulus j. i.e., this transformation may be done by substrating the original mean u_j from all discriminal process in both distribution. Then the mean of the discriminal dispersion of j is now zero ($u_j - u_j = 0$), and the mean of the discriminal dispersion of k is now $u_k - u_j$. This value is also the mean of the difference between all possible discriminal dispersions. To prove this recall the new distribution was created by taking differences between pairs of discriminal process, one from each of the discriminal dispersions.

Let pick a discriminal process with a value s from distribution k and calculate the mean difference between s and all discriminal processes in j. This average will be s,

since the discriminal dispersion of j is symmetric around 0. That is, for every discriminal process with value x there is one with value -x with the same density defined by the discriminal dispersion of j, and their effects cancel. If one repeats this procedure for all discriminal processes in k produces, a symmetric distribution around $u_k - u_j$ will result. Then the mean of the difference of the discriminal dispersion is $u_k - u_j$.

Figure 3 presents a hypothetical distribution of differences, with a mean $u_k - u_j$ and a standard deviation σ_{kj}. The shaded area in the figure 3 indicates the proportion of times the difference $d_k - d_j$ was positive, and the unshaded area indicates the proportion of time $d_k - d_j$ was negative. d_i is an arbitrary discriminal process for stimuli i.

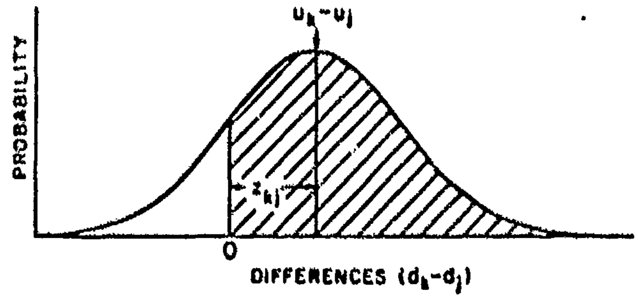

Figure 3. Hypothetical normal distribution of differences between discriminal processes $(d_K - d_j)$. Data obtained by pairing stimuli j and k on many occasions.

The normal density function may be defined by the equation:

$$\phi(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(x-u)^2}{2\sigma^2}\right] \qquad (1)$$

The total area under the curve is 1. By integrating equation 1 the area under any section of the curve may be determined.

$$P(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{x} \exp\left[-\frac{(y-u)^2}{2\sigma^2}\right] dy \qquad (2)$$

The distribution function can not be expressed in closed form in terms of elementary functions. The distribution function is usually tabulated for a normal random variable that has a mean value of zero and a variance of unity (standard normal distribution). It is often designated by $\Phi(x)$ and is defined by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left[-z/2\right]^2 dz \qquad (3)$$

By converting the probability with which k > j (shaded area on figure 2) into a cumulative standard normal distribution, one obtains a standarlized measure of the difference between discriminal processes (uk - uj). Therefore,

$$z_{kj} = \frac{u_k - u_j}{\sigma_{kj}} \qquad (4)$$

where σ_{kj} is the standard deviation of the differences of stimulus pair. The standard deviation of the difference between two normal distribution is

$$\sigma_{kj} = \sqrt{\sigma_j^2 + \sigma_k^2 - 2r_{kj}\sigma_j\sigma_k} \qquad (5)$$

where r_{jk} is the correlation coefficient. For the derivation of equation 5 see appendix A. Rearranging equation 4

$$u_k - u_j = Z_{kj}\sigma_{kj} \quad (6)$$

substituting equation 5 into equation 6, one obtain Thurstone's complete law of comparative judgment

$$u_k - u_j = Z_{kj}\left(\sigma_k^2 + \sigma_j^2 - r_{kj}\sigma_k\sigma_j\right)^{1/2} \quad (7)$$

where u_k and u_j are the mean value for stimuli k and j respectivelyy. σ_k and σ_j are the standard deviation for stimuli k and j. r_{kj} is the correlation coefficient between stimluli k and j. Z_{kj} is the normal deviate corresponding to the theoretical proportion of time stimulus k is judged greater than stimulus j.

The LCJ is not solvable in its complete form, since, regardless of the number of stimuli, there are always, more unknowns than observation equations. For examples, with n stimuli, there are n scale values, n standard deviation, and n(n-1)/2 independent correlation which are unknown. The zero point of the scale can be set arbitrarily at the scale value of one stimulus, and the unit can be taken as one of the standard deviation, leaving 2(n-1) + n(n-1)/2 unknowns. Against this we have only n(n-1)/2 observation equations - one for each independently observable proportion. The number of equations is always 2(n-1) less than the number of

unknowns. Symplifying hypotheses are thus necessary in order to make the law workable.

Thurstone [3] presented five cases of the LCJ. In case 1, the complete form of the LCJ can be used making at least one assumption. The correlation between discriminal deviation is practically constant throughout the stimulus series and for single observer. Case 2 is the same as case 1. The only difference is the use of several observers. Case 3, case 4, and case 5 denote three special sets of equations obtained from various simplifying assumptions. For the measurement definition problem, case 4 of the LCJ was implemented (see next section). Lets now present the approach followed by Thurstone [3] to develop case 4.

In case 4, Thurstone assumes the correlation coefficient is equal to zero and the standard deviation is not subject to gross variation. With those assumptions, the complete form of the LCJ can be simplified so that it becomes linear. If the correlation coefficient is zero (r = 0), using equation 7, the law takes the following form

$$U_k - U_j = Z_{kj} \left(\sigma_k^2 + \sigma_j^2\right)^{1/2} \qquad (8)$$

Assume that

$$\sigma_k = \sigma_j + d \qquad (9)$$

in which d is assume to be smaller than σ_j. Equation 8 becomes

$$U_k - U_j = Z_{kj}\left((\sigma_j + d)^2 + \sigma_j^2\right)^{1/2}$$

$$= Z_{kj}\left(\sigma_j^2 + 2\sigma_j d + d^2 + \sigma_j^2\right)^{1/2}$$

The term d^2 may be dropped if d is small.

$$U_k - U_j = Z_{kj}\left(\sigma_j^2 + 2\sigma_j d + \sigma_j^2\right)^{1/2}$$

$$= Z_{kj}\left(2\sigma_j\right)^{1/2}\left(\sigma_j + d\right)^{1/2} \qquad (10)$$

Let expand the term $(\sigma_j + d)1/2$ in equation 10 , and lets use the first two terms.

$$U_k - U_j = Z_{kj}\left(2\sigma_j\right)^{1/2}\left[(\sigma_j)^{1/2} + \frac{1}{2}(\sigma_j)^{-1/2} d \ \ldots\right]$$

$$= Z_{kj}\left[\sqrt{2}\,\sigma_j + d/\sqrt{2}\right] \qquad (11)$$

Rewriting equation 9 as follow

$$d = \sigma_k - \sigma_j \qquad (12)$$

Substituting equation 12 in equation 11, one obtains

$$U_k - U_j = Z_{kj}\left[\sqrt{2}\,\sigma_j + (\sigma_k - \sigma_j)/\sqrt{2}\right]$$

$$= Z_{kj}\left[\sigma_j/\sqrt{2} + \sigma_k/\sqrt{2}\right] \qquad (13)$$

Equation 13 is the case 4 of the LCJ.

In [4], Torgerson found an approximating equation for the LCJ that is formally identically with the equation 13. He assumed that the c correlation coefficient are all equal and the difference between standard deviation are small. The equation is as follows:

$$U_k - U_j = Z_{kj}\left((1-r)/2\right)^{1/2}(\sigma_k + \sigma_j) \qquad (14)$$

He demonstrated that the assumption of r=0 was unnecessarily restricting. One needs only to assume that the correlation are all equal. For the demotration of equation 14 the reader is referred to reference 4.

C.- The Method of Paired Comparisons

The law of comparative judgment assume that each stimulus has been compared with each other stimulus a large number of times. Hence, the law requires that data of the form "the proportion of times any stimulus k is judged greater than any other stimulus j" are available. The direct method for obtaining empirical estimates of these proportions is known as the method of paired comparisons. This method is essentially a generalization of the two-category case of the method of constant stimuli, where in the method of constant stimuli, each stimulus is compared with a single standard and in paired comparisons each stimulus serves, in turn, as the

standard. In paired comparisons, each stimulus is paired with each other, that means that with n stimuli there are thus n(n-1)/2 pairs. Each pair is presented to the subject, whose task is to indicate which member of the pair appears greater with respect to the attribute to be scaled. The subject must designate one of the pair as greater, and no subject must designate one of the pair as greater, and no equality judgments are allowed. This is consistent with the derivation of the law , wherein the probability of a zero discriminal differences is vanishingly small.

To obtain data from which the proportion may be estimated, a large number of comparison have to be made for each pair of stimuli. There exist three alternatives where the necessary replication might be obtained:

1. having a single subject judge each pair a large number of times,
2. many subjects each judge each pair once, or
3. several subjects each judge each pair several times.

The choice of these alternatives will may depend on the purpose of the experiment, the extent of individual differences, and the nature of the stimuli.

Caution has to be taken in order to use either the first or third alternative that the stimuli should be such that no

extraneous differentiation cues are available to the subject. If the subject can identify the stimulus pairs, there is the possibility that he will base his later judgments on his memory of his earlier judgments of the pair.

In the law of comparative judgment, no explicit provision is made for time or space errors. Nor is there provision for changes in performance due to fatigue or practice effects, or for judgments based in poart on factors other than the relative magnitudes of the discriminal processes. Then, it is necessary to control experimentally the conditions that might introduce these biasing effects. Most of these factors can be controlled in the assignment of the relative positions (spatial or temporal) of the members of each stimulus pair and the order of presentation of the pair themselves. An experiment can be controlled by randomization of relative positions and of orders. This method is not the most efficient one. More efficient methods use counterbalancing procedures. For example, time (or space) errors can be controlled by arranging the members of the pairs so that half the time each stimulus appears first (or to the left, below, etc.) and half the time second (to the right, above, etc.). Perhaps, the best procedure is to counterbalance each pair of stimuli: e.g., with stimulus pair j, k, present j first half the time, k first the other half. Practice or fatigue effects can be controlled by

reversing the order of presentation of the pairs for half of the subject (or trials).

In [4], Togerson presents a list of additional precaution, some of which may or may not be relevant for any given experiment., These additional precaution are:

1. Keeping pairs having one stimulus in common maximally separated in the order of presentation.
2. Arranging pairs so that "correct" responses are approximately evenly divided between first and second memebers of the pairs.
3. Arranging pairs so that there is no detectable systematic pattern of "correct" responses.
4. Arranging pairs so that there is no systematic variation in difficulty of judgment.
5. Varying the order of presentation from trial to trial to eliminate serial learning of a response pattern.

Ross [17] gives a table of the balanced optimal orders for odd numbers of members from five to seventeen. Also, he presented his general method for calculating orders of presentation. His orders are optimal in the sense that a) each stimulus appears first in half the pairs of which it is a member, b)pairs having one stimulus in common are maximally separated in the order of presentation, and c) there is no

detectable pattern of "correct" responses. His orders have the following advantages:

1. They maintain the greatest possible spacing between pairs involving identical members.
2. They are so balanced as to remove time and space erros.
3. They avoid regular repetitions which might have suggestion effects.
4. By repeating the series in reverse order fatigue effects may be balanced out.
5. From these orders for odd-number of members, the optimum even-number orders may be obtained by a simple rule.

In [16], Wherry shows that Ross's optimum lists are not optimum in all senses, and he presented an empirically derived list for seven items which is superior to the list given by Ross [18]. Also, a method is given, whereby any list, arrived at either rationally or empirically, may be rewritten in 8n different ways, by use of 4 step given in [16]. It is shown that 2n of these lists may be combined in such a fashion that fatigue effects are cancelled out.

4.-ANALYTICAL PROCEDURES

The complete form of the law of comparative judgment (LCJ) is

$$U_K - U_j = Z_{kj}(\sigma_k^2 + \sigma_j^2 - 2 r_{kj}\sigma_k\sigma_j)^{1/2} \qquad (15)$$

An experimental test of the complete LCJ has not been conducted because of the problems encountered in determining values for the unknowns, standard deviation and the correlation coefficient between pairs [12]. Simplifying assuptions are usually made to reduce these difficulties (see section 3). The model more widely employed is Thurstone's case V ($r(k,j)=0$ and $\sigma_k = \sigma_j$). Case V assumes that one can ignore the standard deviation associated with individual stimuli because they are constant and their discriminal processes are uncorrelated. That is, knowing the occurrence of a discriminal process from one distribution would not help us predict the discriminal process from another. Because for our case the stimuli are complex the best fit for our observational data is necessary. We select case IV for the fitting of our observational data and propose a method based on the solution of the complete form of the LCJ which has less restrited assumption that case IV. Before describing Case IV and our proposed method, we describe how the observational data is rearrange to be used by either method.

After each of the n(n-1)/2 pairs of stimuli have been presented a large number of times, we have as raw data the number of times each stimulus was judged greater than each other stimulus. These observed frequencies may be arranged in the n x n squared matrix R. The general element r(j,k), which appears at the intersection of the jth row and kth column, denotes the observed number of times stimulus k was judged greater than stimulus j. The diagonal cells of matrix R will ordinarily be left vacant. No comparisons are made between the same stimulus. Since the symmetric cells (e.g., r(2,3) and r(3,2)) sum to the total number of judgments made, the matrix contains n(n-1)/2 independent cells.

Lets construct matrix P from matrix R. The element p(j,k) is obtained by dividing the element r(j,k) by the number of total observation, and it is the observed proportion of times stimulus k was judged greater than stimulus j. Diagonal cells are ,again, ordinarily left vacant. Symmetric cells now sum to unity (e.g., p(2,3) + p(3,2)=1).

After the matrix P is constructed, the basic transformation matrix ,X, is constructed. The element x(j,k) is the unit normal deviate corresponding to the element p(j,k), and may be obtained by referring to a table of areas under the unit normal curve. The element x(j,k) will be

positive for all values of p(j, k) over 0.50, and negative for all values of p(j, k) under 0.50. Proportions of 1.00 and 0.00 cannot be used since the x values corresponding to these proportions are unboundedly large. When such proportions occur, the corresponding cells in matrix X are left vacant. Zeros are entered in the diagonal cells since we can ordinarily assume that here U(k)-U(j) = 0. The matrix is skew-symmetric: that is, the symmetric elements sum to zero, since, e.g., x(2,3)= - x(3,2).

Matrix X contains the sample estimates x(j, k) of the theoretical values found in the equation of the law of comparative judgment. The element x(j, k) is an estimates of the difference (U(k)-U(j)) between scale values of the two stimuli measured in units of the standard deviation of the distribution of discriminal differences . Each independent element of matrix X is an estimate of a value for one equation of the law.

In case IV of the LCJ, the assumptions are that the discriminal dispersion are not subject to gross variation and the correlation term is zero. Assuming these two conditions, a linear equation is obtained (see section 3). The equation is

$$U_k - U_j = Z_{kj}/\sqrt{2} \quad (\sigma_k + \sigma_j) \qquad (16)$$

Togerson [4] demonstrated that the explicit assumption of zero correlation was unnecessarily restricting. He got the same result as Thurstone assuming the correlation term equals for each pair of comparation and small difference between discriminal dispersion. The approximate equation is

$$U_k - U_j = Z_{kj}\left((1-r)/2\right)^{1/2}(\sigma_k + \sigma_j) \qquad (17)$$

where equation 16 and 17 are equal if the correlation term is equal to zero.

Two method have been presented in the literature for the solution of Case IV. The first method was proposed by Thurstone [18]. In this method, the standard deviation must be estimated from the observational data and then the means are calculated. Burros [19] presented an alternative approximation formula for the estimation of the standard deviation. This approximation yields the same value of standard deviation as Thurstone's method, and involves less labor calculation. The second method is proposed by Gibson [20]. His method consists of a least-square solution for case IV. He displayed case IV as a system of homogeneous linear equations for which a least-square solution is presented, using various conditional equation which fix the origin and the unit of measurement. However, the computational labor that would be involved in obtaining a numerical solution is such that it has not yet been applied

to data.

Method A

This method estimates the standard deviation using

$$\sigma_K = B / V_K - 1 \qquad (K=1, \ldots N) \qquad (18)$$

where

$$B = 2N \Big/ \sum_{k=1}^{N} 1/V_K$$
$$V_K = \left[N \sum z_{jK}^2 - \left(\sum z_{jK} \right)^2 \right]^{1/2} \Big/ N \qquad (19)$$

Using the values obtained for the standard deviation, the scale values (means) of the stimuli may be obtained as follows

$$A\,U_K = \left(\sigma_K \sum_{j=1}^{N} z_{jK} + \sum_{j=1}^{N} \sigma_j z_{jK} \right) \Big/ N \qquad K=(1, \ldots N) \qquad (20)$$

where the constant A is an unknown stretching factor. This constant may be equal to the square root of 2 (Thurstone), may be equally to unity, some larger value, or equal to SQRT(2/(1-r)) [4]. See references [18,4,19] for a complete derivation of the method A equations.

Method B

In this method, the scaling (means) and standard deviation are obtained by solving a system of linear equations. Lets write the equations for a four stimuli comparation as follows:

$$
\begin{aligned}
U_1 - U_2 &= Z_{12}\,(\sigma_1 + \sigma_2)/\sqrt{2} \\
U_1 - U_3 &= Z_{13}\,(\sigma_1 + \sigma_3)/\sqrt{2} \\
U_1 - U_4 &= Z_{14}\,(\sigma_1 + \sigma_4)/\sqrt{2} \\
U_2 - U_3 &= Z_{23}\,(\sigma_2 + \sigma_3)/\sqrt{2} \\
U_2 - U_4 &= Z_{24}\,(\sigma_2 + \sigma_4)/\sqrt{2} \\
U_3 - U_4 &= Z_{34}\,(\sigma_3 + \sigma_4)/\sqrt{2}
\end{aligned}
\qquad (21)
$$

These equations constitute a set of six linearly independent homogeneous linear equations in eight unknowns. Lets arbitrarily select a zero point and the unit of measurement as follows:

$$U_1 = 0 \qquad (22)$$

and

$$\sigma_1 = 1 \,. \qquad (23)$$

Lets substitute equations 22 and 23 into equation 21. We get the following set of six linear equations in six unknowns:

$$
\begin{aligned}
-U_2 - Z_{12}\sigma_2/\sqrt{2} &= Z_{12}/\sqrt{2} \\
-U_3 - Z_{13}\sigma_3/\sqrt{2} &= Z_{13}/\sqrt{2} \\
-U_4 - Z_{14}\sigma_4/\sqrt{2} &= Z_{14}/\sqrt{2} \\
U_2 - U_3 - Z_{23}\sigma_2/\sqrt{2} - Z_{23}\sigma_3/\sqrt{2} &= 0 \\
U_2 - U_4 - Z_{24}\sigma_2/\sqrt{2} - Z_{24}\sigma_4/\sqrt{2} &= 0 \\
U_3 - U_4 - Z_{34}\sigma_3/\sqrt{2} - Z_{34}\sigma_4/\sqrt{2} &= 0
\end{aligned}
\qquad (24)
$$

Solving equation [24], a unique solution, except for the origin and the unit of measurement, is possible with four stimuli, while an overdetermined solution will be available for more than four stimuli. In matrix form, equation 24 can be stated as follows:

$$\begin{pmatrix} -1 & 0 & 0 & -z_{12}/\sqrt{2} & 0 & 0 \\ 0 & -1 & 0 & 0 & -z_{13}/\sqrt{2} & 0 \\ 0 & 0 & -1 & 0 & 0 & -z_{14}/\sqrt{2} \\ 1 & -1 & 0 & -z_{23}/\sqrt{2} & -z_{23}/\sqrt{2} & 0 \\ 1 & 0 & -1 & -z_{24}/\sqrt{2} & 0 & -z_{24}/\sqrt{2} \\ 0 & 1 & -1 & 0 & -z_{34}/\sqrt{2} & -z_{34}/\sqrt{2} \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \\ u_4 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{pmatrix} = \begin{pmatrix} z_{12}/\sqrt{2} \\ z_{13}/\sqrt{2} \\ z_{14}/\sqrt{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A \qquad\qquad B \quad = \quad C, \tag{25}$$

Since A is square for four stimuli, the unique solution is

$$B = A^{-1} C. \tag{26}$$

For more than four stimuli, a solution for equation 24 can be obtained from

$$B = (A^T A)^{-1} A^T C. \tag{27}$$

Equation 27 is a least-square solution in the sense that the means and the standard deviation minimize the sum of the squared discrepancies of the entries in the matrix product

AB, from the corresponding entries in C.

If equations 22 and 23 are replaced by equation 28 and 29, one obtains a greater degree of symmetry which will involve more of the unkowns in each of the observation equations.

$$\sum_{i=1}^{N} u_i = 0 \qquad (28)$$

and

$$\sum_{i=1}^{N} \sigma_i = N . \qquad (29)$$

Lets multiply equation 29 by 1/sqrt(2) z(j,k) to obtain

$$\frac{z_{jk}}{\sqrt{2}} \sum_{i=1}^{N} \sigma_i = \frac{N}{\sqrt{2}} z_{jk} . \qquad (30)$$

Adding equation 28 and the appropriate equation 30 to each equations 21 for five stimuli in matrix form:

$$\begin{pmatrix}
2 & 0 & 1 & 1 & 1 & 0 & 0 & z_{12}/\sqrt{2} & z_{12}/\sqrt{2} & z_{12}/\sqrt{2} \\
2 & 1 & 0 & 1 & 1 & 0 & z_{13}/\sqrt{2} & 0 & z_{13}/\sqrt{2} & z_{13}/\sqrt{2} \\
2 & 1 & 1 & 0 & 1 & 0 & z_{14}/\sqrt{2} & z_{14}/\sqrt{2} & 0 & z_{14}/\sqrt{2} \\
2 & 1 & 1 & 1 & 0 & 0 & z_{15}/\sqrt{2} & z_{15}/\sqrt{2} & z_{15}/\sqrt{2} & 0 \\
1 & 2 & 0 & 1 & 1 & z_{23}/\sqrt{2} & 0 & 0 & z_{23}/\sqrt{2} & z_{23}/\sqrt{2} \\
1 & 2 & 1 & 0 & 1 & z_{24}/\sqrt{2} & 0 & z_{24}/\sqrt{2} & 0 & z_{24}/\sqrt{2} \\
1 & 2 & 1 & 1 & 0 & z_{25}/\sqrt{2} & 0 & z_{25}/\sqrt{2} & z_{25}/\sqrt{2} & 0 \\
1 & 1 & 2 & 0 & 1 & z_{34}/\sqrt{2} & z_{34}/\sqrt{2} & 0 & 0 & z_{34}/\sqrt{2} \\
1 & 1 & 2 & 1 & 0 & z_{35}/\sqrt{2} & z_{35}/\sqrt{2} & 0 & z_{35}/\sqrt{2} & 0 \\
1 & 1 & 1 & 2 & 0 & z_{45}/\sqrt{2} & z_{45}/\sqrt{2} & z_{45}/\sqrt{2} & 0 & 0
\end{pmatrix}
\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \end{pmatrix}
=
\begin{pmatrix} N z_{12}/\sqrt{2} \\ N z_{13}/\sqrt{2} \\ N z_{14}/\sqrt{2} \\ N z_{15}/\sqrt{2} \\ N z_{23}/\sqrt{2} \\ N z_{24}/\sqrt{2} \\ N z_{25}/\sqrt{2} \\ N z_{34}/\sqrt{2} \\ N z_{35}/\sqrt{2} \\ N z_{45}/\sqrt{2} \end{pmatrix}$$

$$E \qquad F = G. \qquad (31)$$

For five stimuli the solution to equation 31 is

$$F = E^{-1} G . \qquad (32)$$

For more than five the solution to equation 31 is

$$F = (E^T E)^{-1} E^T G . \qquad (33)$$

In method B, the computational load is quite heavy but this solution has the advantage of providing the best-fitting mean (ranking) and standard deviation values for a set of paired-comparison data.

If the stimuli compared are very complex, the assumptions for case IV are not valid and the solution obtained will not be the best-fitting mean and discriminal dispersion for the observational data. Maybe the best solution will be the use of the complete form of the LCJ. Until now, an experimental test of the complete form of the LCJ has not been conducted because of the problems encountered in determining values for the unknowns ($U_k, U_j, \sigma_k, \sigma_j, r_{kj}$) [12]. Lets now present a method of getting a solution based on least-squares for the complete form of the LCJ assuming the correlation term to be constant (Method C). Equation 34 shows this.

$$U_k - U_j = Z_{kj} (\sigma_k^2 + \sigma_j^2 - 2r\sigma_k\sigma_j)^{1/2} \qquad (34)$$

or

$$Z_{kj} = (U_k - U_j) / (\sigma_k^2 + \sigma_j^2 - 2r\sigma_k\sigma_j)^{1/2} . \qquad (35)$$

where equation 35 is a nonlinear equation. Lets rewrite equation 35 as

$$f(\vec{X}) = (U_K - U_j)/(\sigma_K^2 + \sigma_j^2 - 2r\sigma_K\sigma_j)^{1/2} \tag{36}$$

where

$$\vec{X} = (U_1, U_2, \ldots, U_N, \sigma_1, \sigma_2, \ldots \sigma_N, r)$$

Apply a Taylor expansion to equation 36 and use the first two terms of the series.

$$f(\vec{X}) = f(\vec{X}_0) + \left(\sum_{i=1}^{2N+1} \frac{\partial}{\partial X_i} f(\vec{X}_0)\right)(\vec{X} - \vec{X}_0) \tag{37}$$

or

$$f(\vec{X}) - f(\vec{X}_0) = \left(\sum_{i=1}^{2N+1} \frac{\partial}{\partial X_i} f(\vec{X}_0)\right)(\vec{X} - \vec{X}_0) \tag{38}$$

which in matrix form is,

$$\begin{pmatrix} (f(\vec{X}) - f(X_0))_i \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} \left.\frac{\partial f}{\partial U_1}\right|_{X_0}, \left.\frac{\partial f}{\partial U_2}\right|_{X_0}, & \cdots & \left.\frac{\partial f}{\partial r}\right|_{X_0} \\ & \vdots & \\ & \vdots & \end{pmatrix} (\vec{X} - \vec{X}_0)$$

$$R \quad = \quad A \quad Y \tag{39}$$

where R is a matrix which each element is given by the difference $x(k,j) - f(X_0)$. The size of this matrix is $n(n-1)/2$ rows by 1 column. A is a matrix where each element is a partial derivative of the function $f(X)$ evaluated for X_0. The size of this matrix is $n(n-1)/2$ rows by $2n+1$

columns. Y is a matrix of the unknown variables minus the initial condition X_o. The size of the matrix is 2n + 1 rows by 1 column. Multiply both side of equation 39 by the transpose of matrix A (A').

$$A'R = A'AY \quad (40)$$

Equation 40 is a least squares solution (see appendix B).

If equation 40 is multiply by (A'A) , then we obtain

$$Y = (A'A)^{-1} A'R \quad (41)$$

but

$$Y = \bar{X} - \bar{X}_o$$

then

$$\bar{X} = Y + \bar{X}_o$$

The first N elements of the X are our scaling or ranking of the N-stimulus.

In the next section, the computer implementation for method A is presented.

5.-COMPUTER IMPLEMENTATION

This section presents the algorithm used for the implementation of method A presented as the solution for case IV of the law of comparative judgment (LCJ). Methods B and C have not been implemented yet. In this section, we will propose an algorithm for both methods.

METHOD A

This method is implemented using the subroutine TCIV. Given as an input the basic transformation matrix X which each element is the normal deviate corresponding to the proportion of empirical judgment j>k, and the total number of stimulis used N, the algorithm used to construct this subroutine is as follows:

a) Sum each column of matrix X

$$SX(k) \Rightarrow \sum_{j} Z_{jk} \qquad j,k = 1,2,\dots N.$$

b) Sum the square of each element of matrix X by column

$$SQX(k) \Rightarrow \sum_{j} Z_{jk}^{2} \qquad j,k = 1,2,\dots N.$$

c) Multiply the sum of the squares by the total number of stimuli by column

$$SQXN(K) \Rightarrow N \sum_{j} Z_{jK}^{2} .$$

d) Square the sum of each column of matrix X by column

$$SQSX(K) \Rightarrow \left(\sum_{j} X_{jK}\right)^{2} .$$

e) The square root of the difference of step c and d is obtained by column

$$XNV(K) \Rightarrow (SQXN(K) - SQSX(K))^{1/2} .$$

f) The inverse of step e is obtained by column

$$XIXNV(K) \Rightarrow 1/XNV(K) .$$

g) The sum of the inverse of step e is obtained

$$SUMINV \Rightarrow \sum_{K} XIXNV(K) .$$

h) XNB is obtained

$$XNB \Rightarrow 2N/SUMINV .$$

i) The standard deviation is now obtained by column

$$\sigma(K) \Rightarrow XNB * XINV(K) - 1.$$

j) Check if the sum of standard deviations are equal to N

$$\sum_{K} \sigma(K) = N.$$

k) In this step the sum of the elements of matrix X by row is obtained

$$SRX(j) \Rightarrow \sum_{K} Z_{jK}.$$

l) Multiply each result of step k by the corresponding discriminal dispersion

$$SXRX(j) \Rightarrow SRX(j) * \sigma(j).$$

m) Multiply step a by the discriminal dispersion

$$SXSX(K) \Rightarrow SX(K) * \sigma(K).$$

n) The difference of step m and step l is

obtained

$$XS(k) \Rightarrow SXSX(k) - SXRX(k).$$

o) Divide XS(k) by the total number of stimuli and the scaling for each stimuli is obtained.

$$S(k) \Rightarrow XS(k)/N.$$

p) Check for the sum of the scaling

$$\sum_{k} S(k) \Rightarrow 0.$$

METHOD B

Like method A, the inputs are matrix X and a variable N. The algorithm is as follows:

a) Create matrix E and G.

b) Check for the number of stimuli N.

1) If n is less than 5 stop.

2) If N is no equal 5 go to step c

3) else

1- Get the inverse of matrix E.

2- Multiply the inverse of matrix E by matrix G.

3- Matrix F will contain the ranking and the standard deviation.

4- Go to step h.

c). Get the transpose of E (E').

d) Multiply matrix E' by E and matrix E' by G.

e) Get the inverse of the multiplication of matrix E' by E.

f) Multiply the results of step e by the results obtained in step d for the multiplication of matrix E' by G.

g) The results of step f is the matrix F that contains the ranking and standard deviation.

h) Check if the sum of the first N elements of the matrix F are equal to zero, and the sum of the last N elements are equal to N.

i) Stop.

METHOD C

Like the other two methods, the input will be the matrix X and the total number of stimuli N. This method needs an initial condition. The result obtained from method A or method B can be used as initial condition. If method A is used, the initial condition for the correlation term is set to negative one. For method B, it is set to zero. The algorithm is as follows:

a) Construct matrix A and matrix R.

b). Find the transpose of matrix A (A').

c) Find the inverse of the product A'A.

d) Multiply the transpose A′ by R.

e) Multiply the results of step c by step d.

f) Check the elements of the matrix resulting from step e. If all the elements are less than a giving accuracy go to g else go to h.

g) The results is the sum of the results of step e plus the initial condition. Stop.

h)Sum the results of step e plus the initial condition. This result will be the new initial condition. Go to a.

In appendix C, the listing of the program for method A is presented.

6. -RESULTS

This section presents the results obtained for two given data sets. The two data sets were obtained from references [4, 18]. The results obtained by our program are compared with the results already published in references [4, 18].

The first data set were taken from reference [4]. Table 1 shows the data set. The results given by Torgerson [4] are showed in table 2. Table 3 shows our results, which agree with the results given in table 2.

MATRIX X

		Stimuli k 1	2	3	4	5
Stimuli j	1	0.0000	0.2778	0.6818	1.2500	1.2500
	2	−0.2778	0.0000	0.5000	1.0714	1.1364
	3	−0.6818	−0.5000	0.0000	0.2778	0.5769
	4	−1.2500	−1.0714	−0.2778	0.0000	0.5000
	5	−1.2500	−1.1364	−0.5769	−0.5000	0.0000

Table 1. Data Set Given by Torgerson [4] for his Illustrative Example.

	Stimuli 1	2	3	4	5
STD. DEV.	1.0634	0.8490	1.2165	0.5856	1.2854
SCALING	−1.417	−0.893	0.129	0.633	1.548

Table 2. Results Given by Torgerson [4].

MATRIX X

0.0000	0.2778	0.6818	1.2500	1.2500
-0.2778	0.0000	0.5000	1.0714	1.1364
-0.6818	-0.5000	0.0000	0.2778	0.5769
-1.2500	-1.0714	-0.2778	0.0000	0.5000
-1.2500	-1.1364	-0.5769	-0.5000	0.0000

```
STD. DEV. ( 1)=    1.063384
STD. DEV. ( 2)=    0.849030
STD. DEV. ( 3)=    1.216482
STD. DEV. ( 4)=    0.585505
STD. DEV. ( 5)=    1.285470
```

```
SCALING ( 1)=  -1.416595
SCALING ( 2)=  -0.872860
SCALING ( 3)=   0.128641
SCALING ( 4)=   0.632680
SCALING ( 5)=   1.548135
```

Table 3. Results Obtained by our Program TCIV.

The second data set were taken from reference [18]. These data were obtained experimentally by Thurstone. These data consists of thirteen nationalities or races. Each one of this nationality was paired with every other nationality. The number of pair was 78. These pairs were arranged in a

printed schedule and were submitted to 250 high school children in Chicago. The instruction were given with the following printed schedule.

> This is an experimental study of attitudes toward races and nationalities. You are asked merely to underline the one nationality, or race, of each pair that you would rather associate with. For example, the first pair is:
>
> ENGLISHMAN - SOUTH AMERICANS
>
> If in general, you prefer to associate with ENGLISHMAN rather than with SOUTH AMERICANS, underline ENGLISHMAN. If you prefer, in general, to associate with SOUTH AMERICANS, underline SOUTH AMERICANS. If you find it difficult to decide for any pair, simply underline one of them anyway. If two nationalities are about equally well liked, they will have about the same number of underlinings in all of the papers. Be sure to underline one of each pair even if you have to make a sort of guess.

Table 4 shows the experimental proportion. It shows the proportion of the subjects who preferred each nationality at the top of the table, to each nationality at the side of the table. For example, the proportion of subjects who preferred ENGLISHMAN to SOUTH AMERICANS was .935. The proportion of subjects who preferred SOUTH AMERICANS to ENGLISHMAN was .065, since intermediate categories of judgment were not allowed. Table 5 shows the X-values, and Table 6 shows the results for the standard deviation and the scaling value for each nationality. Table 7 shows the results obtained by our program. The standard deviation results agree with table 6. The scaling results are different, because our stretching factor A is equal to one. If one divides our scaling results by $1/\sqrt{2}$, $A=1/\sqrt{2}$, one obtains the same results as table 6.

EXPERIMENTAL PROPORTIONS

		1 Eng.	2 Ca.	3 Fr.	4 Ir.	5 Sc.	6 Sw.	7 Ge.	8 Ho.	9 Sp.	10 Be.	11 S.A.	12 Jew	13 It.
1......	Eng.	....	.388	.218	.324	.165	.221	.227	.162	.144	.103	.065	.155	.066
2......	Ca.	.612	....	.457	.406	.280	.260	.297	.102	.201	.100	.088	.180	.073
3......	Fr.	.782	.543	....	.541	.500	.370	.380	.255	.184	.214	.149	.192	.081
4......	Ir.	.676	.594	.459	....	.361	.387	.378	.253	.273	.243	.162	.221	.128
5......	Sc.	.835	.720	.500	.639	....	.400	.409	.268	.249	.258	.262	.223	.128
6......	Sw.	.779	.740	.630	.613	.600	....	.471	.444	.377	.317	.318	.229	.228
7......	Ge.	.773	.703	.620	.622	.591	.529	....	.347	.391	.325	.343	.253	.152
8......	Ho.	.838	.898	.745	.747	.732	.556	.653	....	.471	.432	.389	.263	.310
9......	Sp.	.856	.799	.816	.727	.751	.623	.609	.529	....	.510	.422	.289	.217
10......	Be.	.897	.900	.786	.757	.742	.683	.675	.568	.490	....	.461	.360	.271
11......	S.A.	.935	.912	.851	.838	.738	.682	.655	.611	.578	.539	....	.420	.320
12......	Jew	.845	.820	.808	.779	.777	.771	.747	.737	.711	.640	.580	....	.524
13......	It.	.934	.927	.919	.872	.872	.772	.848	.690	.783	.729	.680	.476	....

Table 4. Experimental Proportion Given by Thurstone [18].

X-VALUES

	1 Eng.	2 Ca.	3 Fr.	4 Ir.	5 Sc.	6 Sw.	7 Ge.	8 Ho.	9 Sp.	10 Be.	11 S.A.	12 Jew	13 It.
1 Eng....	.00	−.28	−.78	−.46	−.97	−.77	−.75	−.99	−1.06	−1.26	−1.51	−1.02	−1.51
2 Ca.....	.28	.00	−.11	−.24	−.58	−.64	−.53	−1.27	−.84	−1.28	−1.35	−.92	−1.45
3 Fr.....	.78	.11	.00	.10	.00	−.33	−.31	−.66	−.90	−.79	−1.04	−.87	−1.40
4 Ir......	.46	.24	−.10	.00	−.36	−.29	−.31	−.67	−.60	−.70	−.99	−.77	−1.14
5 Sc......	.97	.58	.00	.36	.00	−.25	−.23	−.62	−.68	−.65	−.64	−.76	−1.14
6 Sw.....	.77	.64	.33	.29	.25	.00	−.07	−.14	−.31	−.48	−.47	−.74	−.75
7 Ge....	.75	.53	.31	.31	.23	.07	.00	−.39	−.28	−.45	−.40	−.67	−1.03
8 Ho....	.99	1.27	.66	.67	.62	.14	.39	.00	−.07	−.17	−.28	−.63	−.50
9 Sp.....	1.06	.84	.90	.60	.68	.31	.28	.07	.00	.03	−.20	−.56	−.78
10 Be.....	1.26	1.28	.79	.70	.65	.48	.45	.17	−.03	.00	−.10	−.36	−.61
11 S.A....	1.51	1.35	1.04	.99	.64	.47	.40	.28	.20	.10	.00	−.20	−.[illegible]7
12 Jew....	1.02	.92	.87	.77	.76	.74	.67	.63	.56	.36	.20	.00	.06
13 It......	1.51	1.45	1.40	1.14	1.14	.75	1.03	.50	.78	.61	.47	−.06	.00

Table 5. X-Values given by Thurstone [18].

	SCALING	S.D.
1 English	1.4050	1.3121
2 Canadian	.8718	.8295
3 French	.4902	.7062
4 Irish	.6159	1.1791
5 Scotch	.2828	.7015
6 Swede	.1029	1.0837
7 German	.1298	1.0122
8 Hollander	−.2573	.7634
9 Spaniard	−.2805	.8224
10 Belgian	−.4229	.7646
11 South American	−.5686	.7170
12 Jew	−1.2540	2.1167
13 Italian	−1.1151	.9918

Table 6. Results Given by Thurstone [18].

```
STD.  DEV.  ( 1)=   1.312225
STD.  DEV.  ( 2)=   0.829539
STD.  DEV.  ( 3)=   0.705908
STD.  DEV.  ( 4)=   1.179264
STD.  DEV.  ( 5)=   0.701411
STD.  DEV.  ( 6)=   1.084149
STD.  DEV.  ( 7)=   1.012095
STD.  DEV.  ( 8)=   0.763156
STD.  DEV.  ( 9)=   0.821752
STD.  DEV.  (10)=   0.764467
STD.  DEV.  (11)=   0.716604
STD.  DEV.  (12)=   2.117647
STD.  DEV.  (13)=   0.991697
```

```
SCALING ( 1)=   1.987375
SCALING ( 2)=   1.233057
SCALING ( 3)=   0.673138
SCALING ( 4)=   0.871243
SCALING ( 5)=   0.379726
SCALING ( 6)=   0.145585
SCALING ( 7)=   0.183501
SCALING ( 8)=  -0.363854
SCALING ( 9)=  -0.396541
SCALING (10)=  -0.578002
SCALING (11)=  -0.804074
SCALING (12)=  -1.774290
SCALING (13)=  -1.577064
```

Table 7. Results Obtained by our Program TCIV.

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APPENDIX A

In this appendix, the standard deviation of the difference between two normal random variables is found.

Assume that X and Y are two normal random variables. The expected value can be written as follows:

$$E\left[(X-Y)^2\right] = E\left[X^2 - 2XY + Y^2\right] \qquad (1)$$

$$= \overline{X^2} - 2\overline{XY} + \overline{Y^2}. \qquad (2)$$

The variance for a random variable is defined as

$$\sigma_X^2 = E\left[(X-U_X)^2\right]$$

$$= E\left[X^2\right] - 2U_X E\left[X\right] + U_X^2$$

$$\sigma_X^2 = \overline{X^2} - U_X^2. \qquad (3)$$

Using the same procedure,

$$\sigma_Y^2 = \overline{Y^2} - U_Y^2. \qquad (4)$$

If one substitutes into equation (2) with (3) and (4) then,

$$E\left[(X-Y)^2\right] = \sigma_X^2 + U_X^2 + \sigma_Y^2 + U_Y^2 - 2\overline{XY}$$

$$= \sigma_X^2 + \sigma_Y^2 + U_X^2 + U_Y^2 - 2\overline{XY}. \qquad (5)$$

The last term in equation (5) is the correlation term and can be defined as

$$E[XY] = \overline{XY}.$$

If X and Y have nonzero means, then it is frequently more convenient to find the correlation by substracting the mean values.

$$E[(X-U_X)(Y-U_Y)] = \overline{(X-U_X)(Y-U_Y)}$$

This is known as the covariance.

To express the degree to which two random variables are correlated without regard to the magnitude of either one, then the correlation coefficient or normalized covariance is the appropriate quantity. It is defined as

$$\rho = E\left[\left(\frac{X-U_X}{\sigma_X}\right)\left(\frac{Y-U_Y}{\sigma_Y}\right)\right] = E[st]$$

$$\therefore \quad s = \frac{X-U_X}{\sigma_X} \qquad t = \frac{Y-U_Y}{\sigma_Y}$$

$$X = s\sigma_X + U_X \qquad Y = t\sigma_Y + U_y \, ,$$

hence

$$\begin{aligned}
\overline{XY} &= E[XY] = E[(s\sigma_X + U_X)(t\sigma_Y + U_y)] \\
&= E[st\sigma_X\sigma_Y + s\sigma_X U_Y + t\sigma_Y U_X + U_Y U_X] \\
&= E[st]\sigma_X\sigma_Y + E[(X-U_X)U_Y] + E[(Y-U_Y)U_X] + E[U_Y U_X] \\
&= \rho\sigma_X\sigma_Y + E[XU_Y - U_X U_Y] + E[YU_X - U_Y U_X] + E[U_Y U_X] \\
&= \rho\sigma_X\sigma_Y + U_X U_Y - U_X U_Y + U_Y U_X - U_Y U_X + U_Y U_X
\end{aligned}$$

$$\overline{XY} = \rho\sigma_X\sigma_Y + U_X U_Y \, . \qquad (6)$$

Substitute equation (6) into equation (5)

$$E[(X-Y)^2] = \sigma_X^2 + \sigma_Y^2 + U_X^2 + U_Y^2 - 2(\rho\sigma_X\sigma_Y + U_X U_Y).$$

$$= \sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y + U_X^2 - 2U_X U_Y + U_Y^2 .$$

The last three terms are just The square of The mean of (X-Y). Then,

$$E[(X-Y)^2] = \sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y + (U_X - U_Y)^2$$

$$= \sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y + (E(X-Y))^2 .$$

In general

$$\sigma^2 = \bar{X}^2 - U^2 , \qquad (8)$$

and from equation (7),

$$\sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y = E[(X-Y)^2] - (E[X-Y])^2. \qquad (9)$$

Comparing equations (8) and (9), the left hand term is the variance for the substraction of two random variables.

$$\sigma_{XY}^2 = \sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y .$$

The standard deviation is the square root of the variance. Then,

$$\sigma_{XY} = (\sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y)^{1/2} . \qquad (10)$$

APPENDIX B

In this appendix, it is demonstrated that A'R=A'AX implies a least square solution

$$A'R = A'AX \Rightarrow \sum (\dot{z}_{Kj} - z_{Kj})^2$$

If

$$\dot{z}_{Kj} - z_{Kj} = \epsilon_{Kj}$$

or

$$(\dot{z}_{Kj} - z_{Kj})^2 = \epsilon_{Kj}^2$$

where ϵ_{Kj} is an error term.

In matrix form

$$\begin{pmatrix} \dot{z}_{Kj} - z_{Kj} \\ \vdots \\ \vdots \end{pmatrix} = \epsilon .$$

and

$$(\dot{z}_{Kj} - z_{Kj}, \ldots\ldots) = \epsilon'$$

so,

$$(\dot{z}_{Kj} - z_{Kj}, \ldots) \begin{pmatrix} \dot{z}_{Kj} - z_{Kj} \\ \vdots \end{pmatrix} = \epsilon'\epsilon$$

$$\begin{pmatrix} (\dot{z}_{Kj} - z_{Kj})^2 \\ \vdots \\ \vdots \end{pmatrix} = \epsilon'\epsilon$$

Let the following equation be a system of observation equations

$$Ax = R .$$

Lets rewrite this as

$$Ax - R = \epsilon$$

where ϵ is a vector of error ϵ_i. The principle of least squares postulates that the most satisfactory values of the x(i) are those for which the sum of the squares of the errors are a minimum. This sum is simply $\epsilon'\epsilon$,

$$\epsilon'\epsilon = (x'A' - R')(Ax - R).$$

The minimal condition is obtained by partially differentiating the above equation with respect to the vector X and halving. We obtain

$$A'Ax = A'R.$$

APPENDIX C

TMDS SUBROUTINE TCIV

LANGUAGE: Fortran VII, using TMDS subroutine, for Perkin-Elmer 8/32.

PURPOSE : To rank the relative visual differences among a set of texture pairs.

INPUT/OUTPUT: Respectively, the samples estimates x(j,k) contained in a matrix X, a single interger value which represents the total number of texture pairs and the standard deviation of each texture pairs.

USAGE : CALL TCIV (X,SCALE,SIGMA,N)

X - input the samples estimates

SCALE - output ranking of texture pairs.

SIGMA - output discriminal dispersion of texture pair.

N - input total number of texture pairs.

PROGRAM LOGIC : The rank of the relative differences among a set of texture pairs is calculated using case IV of the LCJ. The standard deviation is calculated using

$$\sigma_K = B/V_K - 1 \quad (K=1,2,\ldots N)$$

$$B = 2N / \sum_{K=1}^{N} 1/V_K$$

$$V_K = \left(N \sum Z_{jK}^2 - \left(\sum Z_{jK}\right)^2\right)^{1/2}/N.$$

The scale value of the texture pair is calculated using

$$AV_K = \left(\sigma_K \sum_{j=1}^{N} Z_{jK} + \sum_{j=1}^{N} \sigma_j Z_{jK}\right)/N \quad K=1,2,\ldots N$$

where A= 1 and correspond to a correlation of r= -1.

ILENAME = LSU1:TCIV.FTN

```
C
C
C     TEXTURE MEASUREMENT DEFINITION SYSTEM
C
C     REMOTE SENSING AND IMAGE PROCESSING LABORATORY
C
C     ELECTRICAL AND COMPUTER ENGINEERING DEPARTMENT
C
C     LOUISIANA STATE UNIVERSITY. BATON ROUGE.   70803
C
C---------------------------------------------------------------------------
C
C     PROGRAM  TCIV
C
C
C     PURPOSE
C
C          TO RANK THE RELATIVE VISUAL DIFFERENCE AMONG A SET OF TEXTURE
C
C          PAIRS.
C
C
C     METHOD
C
C          THURSTONES CASE IV OF THE LAW OF COMPARATIVE JUDGMENT.
C
C
C     DESCRIPTION OF PARAMETERS
C
C          X     - BASIC TRANSFORMATION MATRIX. EACH ELEMENT OF THE MATRIX
C                  IS THE NORMA DEVIATE CORRESPONDING TO THE PROPORTION
C                  OF EMPIRICAL JUDGMENT J > K.
C
C
C          SIGMA- ONE DIMENSION ARRAY CONTAINING THE DISCRIMINAL
C                  DISPERSIONS.
C
C
C          SCALE- ONE DIMENSION ARRAY CONTAINING THE MEANS OR SCALE VALUES
C
C
C          N     - TOTAL NUMBER OF TEXTURES PAIRS.
C
C
C     REMARK
C
C          THIS PROGRAM IMPLEMENT THE CASE IV OF THE LCJ. THE INPUT
C
C          TO  THIS PROGRAM IS THE MATRIX X AND THE TOTAL NUMBER OF
C
C          TEXTURE PAIRS. THE OUTPUT IS THE SCALING AND THE DISCRIMINAL
C
C          DISPERSION OF THE TEXTURE PAIRS.
C
C
C     AUTHOR
C
```

```
C          R. E. VASQUEZ-ESPINOSA
C
C
C     DATE
C
C          SEPTEMBER 20, 1982
C
C
C     REFERENCE
C
C          THE MEASUREMENT OF VALUES  BY L. L. THURSTONE.
C
C
C ------- ------ ------ ------------------ -------------------------------------
      SUBROUTINE TCIV(X,N,SIGMA,SCALE)
      DIMENSION X(N,N),SIGMA(N),SCALE(N),SUMXCOL(500),SUMSQ(500),
     *VNK(500),XINVNK(500)
C SUM THE ELEMENT OF EACH COLUMMN AND KEEP THE RESULTS BY COLUMN
C SUM THE SQUARE OF ELEMENT OF COLUMN AND KEEP THE RESULTS BY COLUMN
C
      DO 10 K=1,N
         SUMXCOL(K)=0.0
         SUMSQ(K)=0.0
         DO 10 J=1,N
            SUMXCOL(K)=SUMXCOL(K) + X(J,K)
            SUMSQ(K)=SUMSQ(K) + X(J,K)**2
   10 CONTINUE
      SINV=0.0
      DO 20 K=1,N
         VNK(K)=SQRT(N*SUMSQ(K) - SUMXCOL(K)**2)
         XINVNK(K)= 1.0/ VNK(K)
         SINV=SINV + XINVNK(K)
   20 CONTINUE
      XNB= 2.0*N/SINV
      DO 30 K=1,N
         SIGMA(K)=XNB*XINVNK(K) - 1.0
   30 CONTINUE
C
C LET FIND THE SCALING
C
      DO 40 K=1,N
         SUMXCOL(K)=0.0
         DO 50 J=1,N
            SUMXCOL(K)=SUMXCOL(K) + X(J,K)
   50    CONTINUE
         SUMXCOL(K)=SIGMA(K) * SUMXCOL(K)
   40 CONTINUE
      DO 60 K=1,N
         SUMSQ(K)=0.0
         DO 70 J=1,N
            SUMSQ(K)=SUMSQ(K) + SIGMA(J)*X(J,K)
   70    CONTINUE
         SCALE(K)=(SUMXCOL(K) + SUMSQ(K))/N
   60 CONTINUE
      RETURN
      END
```